1 Additional Exercises: Finite Group of Motions

1. Let D_n denote the dihedral group. Express the product $x^2yx^{-1}y^{-1}x^3y^3$ in the form x^iy^j in D_n.
2. List all the subgroups of $D_4$and determine which are normal.
3. Find all proper normal subgroups and identify the quotient groups of the groups D_{13} and D_{15}.
4. Prove any discrete group G consisting of rotations about the origin is cyclic and is generated by ρ_θ where θ is the smallest angle of rotation in G.
   (a) Advanced Calculus students, or others interested in the completeness property of the real numbers, may wish to prove that any discrete group G consisting of rotations about the origin really does have a smallest angle of rotation.
5. Let G be a subgroup of M that contains rotations about two different points. Prove algebraically that G contains a translation.
6. Determine the point group for each of the patterns depicted in the figure on the handout labelled "Extra Exercise: Finite Group of Motions #6."
7. Prove that every discrete subgroup of O is finite.
8. Prove the group of symmetries of the frieze pattern

   $$\cdots \text{EEEEEEEE} \cdots$$

   is isomorphic to the direct product $C_2 \times C_\infty$ of a cyclic group of order 2 and and infinite cyclic group.
9. Let G be the group of symmetries of the frieze pattern

   $$\cdots \subset \supset \subset \supset \subset \supset \cdots$$

   (a) Determine the point group $\overline{G}$ of G.
   (b) For each element $\overline{g}$ of $\overline{G}$, and each element g of G which represents $\overline{g}$, describe the action of g geometrically.
   (c) Let H be teh subgroup of translations in G. Determine $[G : H]$.
10. Let G be a discrete group in which every element is orientation-preserving. Prove the point group $\overline{G}$ is a cyclic group of rotations and there is a point p in the plane such that the set of group elements which fix p is isomorphic to $\overline{G}$.
11. Let Ndenote the group of rigid motions of the line $l = R^1$.Some elements of N are

    $$t_a : t_a(x) = x + a \text{ and } s : s(x) = -x.$$

(a) Show that $\{t_a, t_a s\}$ are all of the elements of N,and describe their actions on l geometrically. [Note that $|N|$ is infinite since there is a distinct t_a for each real number a.]

(b) Compute the products $t_a t_b$, st_a, ss.

(c) Find all discrete subgroups of N which contain a translation. It will be convenient to choose your origin and unit length with reference to the particular subgroup. Prove your list is complete.

12. Prove if the point group of a lattice group G is C_6, then $L = L_G$ is an equilateral triangular lattice, and G is the group of all rotational symmetries of L about the origin.

13. Prove if the point group of a lattice group G is D_6, then $L = L_G$ is an equilateral triangular lattice, and G is the group of all symmetries of L.